UNIVERSITY OF CAMBRIDGE

NATURAL SCIENCES TRIPOS Part II

Friday 30 May 2014 1.30 pm to 3.30 pm

PHYSICS (7)
PHYSICAL SCIENCES: HALF SUBJECT PHYSICS (7)

QUANTUM CONDENSED MATTER PHYSICS

Candidates offering this paper should attempt a total of **three** *questions. The questions to be attempted are* **1**, **2** *and* **one** *other question.*

The approximate number of marks allocated to each question or part of a question is indicated in the right margin. This paper contains **four** *sides, and is accompanied by a handbook giving values of constants and containing mathematical formulae which you may quote without proof.*

STATIONERY REQUIREMENTS
2 × 20 Page Answer Book
Rough workpad
Yellow master coversheet

SPECIAL REQUIREMENTS
Mathematical Formulae handbook
Approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator.

QUANTUM CONDENSED MATTER PHYSICS

1 *Attempt* **all** *parts of this question. Answers should be concise and relevant formulae may be assumed without proof.*

(a) Calculate the plasma frequency, $\omega_{\rm p} = \sqrt{ne^2/(m\varepsilon_0)}$, for a typical metal. What is the physical significance of this quantity? [4]

(b) Consider three atoms arranged in an equilateral triangle. The 'on-site' energy of an electron on each atom is $\langle n|H|n\rangle = E_0$, where $|n\rangle$ denotes the spatial wave function of an electron localised on the $n^{\rm th}$ atom, with $n = 1, 2, 3$. The hopping term between neighbouring atoms is $\langle n|H|n+1\rangle = t < 0$. What are the ground-state spatial wave function and energy of a single electron in this ring? [4]

(c) Consider N conduction electrons in a volume V, neglect interactions between electrons with the same spin, and assume that the energy per electron due to repulsion between electrons of opposite spin is $UN_\uparrow N_\downarrow/N^2$. For U above some critical value, $U_{\rm c}$, a ferromagnetic state has a lower energy than an unpolarised one with $N_\uparrow = N_\downarrow = N/2$. Find the scaling of $U_{\rm c}$ with V and N. [4]

2 *Attempt this question. Credit will be given for well-structured and clear explanations, including appropriate diagrams and formulae. Detailed mathematical derivations are not required.*

Write brief notes on **two** of the following: [13]

(a) covalent and ionic bonds;

(b) DC conductivity and the Hall effect within the Drude model;

(c) the nearly-free-electron and tight-binding models, emphasising how different energy bands arise in each case.

3 *Attempt* **either** *this question* **or** *question 4.*

In a particular material with 'spin-orbit coupling', the energy of the conduction electrons depends on both motional and spin degrees of freedom. Using plane-wave basis states $|\boldsymbol{k}\sigma\rangle$, with wavevector $\boldsymbol{k}$ and spin-state σ quantised along the z-direction, the general state has the form

$$|\psi_{\boldsymbol{k}}\rangle = \alpha\,|\boldsymbol{k}\uparrow\rangle + \beta\,|\boldsymbol{k}\downarrow\rangle \ ,$$

and the Hamiltonian can be expressed as

$$H = \begin{pmatrix} \hbar^2 k^2/(2m) & \lambda(-k_y - \mathrm{i}k_x) \\ \lambda(-k_y + \mathrm{i}k_x) & \hbar^2 k^2/(2m) \end{pmatrix} ,$$

where λ is a constant and m is the electron effective mass.

Show that the energy dispersion in this material has two bands, which for $k_z = 0$ take the form

$$E_\pm = \frac{\hbar^2}{2m}\left(k^2 \pm q_\mathrm{R}\,|\boldsymbol{k}|\right) ,$$

and express q_R in terms of λ, $\hbar$ and m. [5]

Sketch the dispersion of the spin-split bands $E_+(\boldsymbol{k})$ and $E_-(\boldsymbol{k})$ along $\boldsymbol{k} = (k_x, 0, 0)$, indicating the minimal value of E_- and the value(s) of k_x for which it occurs. [4]

For $\boldsymbol{k}$ in the $k_z = 0$ plane, sketch the Fermi surfaces for three different values of the Fermi energy: (i) $-\hbar^2 q_\mathrm{R}^2/(8m) < E_\mathrm{F} < 0$, (ii) $E_\mathrm{F} = 0$, and (iii) $E_\mathrm{F} > 0$. In each case clearly indicate which $\boldsymbol{k}$ states are unoccupied, singly occupied, and doubly occupied. [6]

Measurements of the Fermi surface cross-section in this material, for $k_z = 0$ and $E_\mathrm{F} > 0$, reveal two Fermi wavevectors: $k_1 = 0.11\text{Å}^{-1}$ and $k_2 = 0.01\text{Å}^{-1}$. Find the value of q_R. [4]

For $k_z = 0$, the spin eigenstates of H in the $E_-(\boldsymbol{k})$ band are

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} (k_y + \mathrm{i}k_x)/|\boldsymbol{k}| \\ 1 \end{pmatrix} .$$

Deduce the spin eigenstates in the $E_+(\boldsymbol{k})$ band and sketch how the spin direction varies with the direction of $\boldsymbol{k}$ on the two Fermi surfaces, $|\boldsymbol{k}| = k_1$ and $|\boldsymbol{k}| = k_2$. [6]

[*You may use the following eigenstates and eigenvalues of the Pauli matrices:*

$$\sigma_x \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \quad , \quad \sigma_y \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm \mathrm{i} \end{pmatrix} = \pm\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \pm \mathrm{i} \end{pmatrix} \quad .$$
]

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4 *Attempt* **either** *this question* **or** *question 3.*

Show that the intrinsic carrier concentration n_i in a non-degenerate semiconductor is given by

$$n_\mathrm{i}^2 = np = \frac{1}{2}\,(m_\mathrm{e}m_\mathrm{h})^{3/2}\left(\frac{k_\mathrm{B}T}{\pi\hbar^2}\right)^3 \exp\left(-\frac{E_\mathrm{g}}{k_\mathrm{B}T}\right),$$

where m_e and m_h are the electron and hole effective masses, E_g is the band gap energy, and n and p are the electron and hole concentrations. [8]

[*You may find one of the following results useful:* $\int_0^\infty \sqrt{x}\,\mathrm{e}^{-x}\,\mathrm{d}x = \sqrt{\pi}/2$, $\int_0^\infty x^2\,\mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}/4$.]

Make an annotated sketch of the bending of the conduction and valence bands, and of the electron and hole concentrations, across an unbiased *p–n* junction. [5]

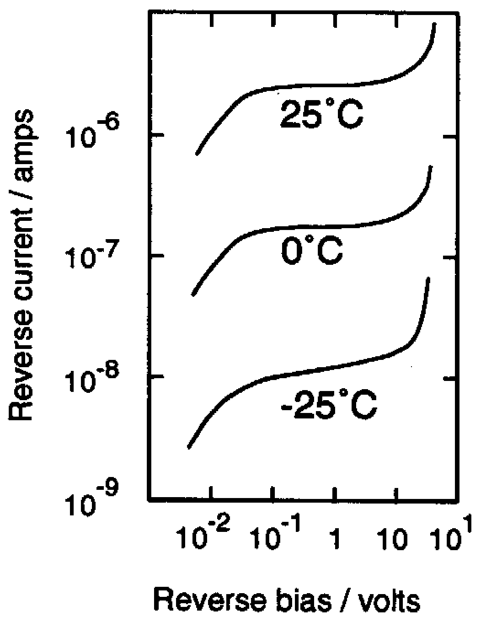
The current in a reverse-biased *p–n* junction diode with an applied voltage V follows the diode equation:

$$I = I_\mathrm{sat}\left[1 - \exp\left(-\frac{eV}{k_\mathrm{B}T}\right)\right],$$

where the saturation current I_sat is proportional to n_i^2. Identify the origins of the two contributions to I. [4]

The diagram below shows measurements of the current across a germanium *p–n* diode under reverse bias. Explain the form of the curves and their temperature dependence. [4]

Deduce the value of the band gap in germanium. [4]



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